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NUMERICAL METHODS FOR STIFF, QUADRATIC AND JOSEPHSON INTERFEROM--ETC(U)
AUG 80 W LINIGER, F ODEH F49620-77-C-0088

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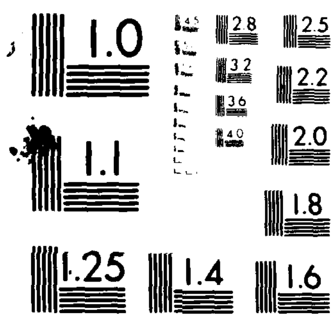

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Stiff equations, Multistep and one-leg methods. Variable steps. Variable coefficients. Stability and contractivity. Josephson interferometer equations. Running solutions. Resonances.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The two-parameter class of all A-contractive two-step second-order formulas was derived for arbitrary step ratios. Similarly, the one-parameter class of all A-constractive two-step second-order formulas was derived for arbitrary step ratios. A specific A-contractive formula was derived which minimizes a measure of the global truncation error. The one-leg implementation of the A-contractive formulas provides A-stable methods for $\dot{x} = \lambda(t)x$ for any $\lambda(t)$, Re $\lambda(t) \leq 0$ and any step sequence $\{h_n\}$. It was shown that

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among all A-stable two-step second-order formulas for uniform steps, the A-constractive formulas are the only ones for which A-stability is conserved under any sufficiently small perturbation of the uniformity of the steps.

An efficient numerical study, as well as perturbation analysis of the solutions to the two-junction interferometer was carried out. We obtained the dependence of the resonant current on the interferometer's material parameters as well as the previously undetected subharmonic resonances. Also the shapes of the phase functions, both near and far from resonance, were found.

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Numerical Methods for Stiff, Quadratic and Josephson Interferometer Differential Equations.

AFOSR-TR- 80-0952

ABSTRACT

The two-parameter class of all A_0-contractive two-step second-order formulas was derived for arbitrary step ratios. Similarly, the one-parameter class of all A-contractive two-step second-order formulas was derived for arbitrary step ratios. A specific A-contractive formula was derived which minimizes a measure of the global truncation error. The one-leg implementation of the A-contractive formulas provides A-stable methods for $\dot{x} = \lambda(t)x$ for any $\lambda(t)$, Re $\lambda(t) \leq 0$ and any step sequence $\{h_n\}$. It was shown that, among all A-stable two-step second-order formulas for uniform steps, the A-contractive formulas are the only ones for which A-stability is conserved under any sufficiently small perturbation of the uniformity of the steps.

An efficient numerical study, as well as perturbation analysis, of the solutions to the two-junction interferometer was carried out. We obtained the dependence of the resonant current on the interferometer's material parameters as well as the previously undetected subharmonic resonances. Also the shapes of the phase functions, both near and far from resonance, were found.

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1. RESEARCH OBJECTIVES

I. Derive reliable and efficient variable-step methods for solving variable coefficient and nonlinear differential equations.

II. Use analytical and numerical methods for determining the responses of interferometers and other devices involving Josephson junctions.

2. STATUS OF RESEARCH EFFORT

I. Stiff Equations

a. Background

On a non-uniform grid $\{t_n\}$, $t_n = \sum_{\nu=1}^{n} h_\nu$, $n = 1, 2, \ldots$, $h_n = t_n - t_{n-1}$, the linear multistep formulas written in operator form are

$$\rho_n x_n - h_n \sigma_n \dot{x}_n = 0, \tag{1}$$

where $\rho_n x_n := \sum_{j=0}^{k} \alpha_{j,n} x_{n-k+j}$, $\sigma_n \dot{x}_n := \sum_{j=0}^{k} \beta_{j,n} \dot{x}_{n-k+j}$. For uniform steps, $\rho x_n = E^{-k}\rho(E)x_n$ and $\sigma \dot{x}_n = E^{-k}\sigma(E)\dot{x}_n$, where $Ex_n := x_{n+1}$ and $\rho(\zeta) = \Sigma\, \alpha_j \zeta^j$, $\sigma(\zeta) = \Sigma\, \beta_j \zeta^j$ are the familiar polynomials. The formula (1) is normalized by

$$\sigma_n 1 = \sum_{j=0}^{k} \beta_{j,n} = 1. \tag{2}$$

In order to solve the non-linear system $\dot{x} = f(t, x)$, the formula (1) can be implemented either as a linear multistep (MS) method.

$$\rho_n x_n - h_n \sigma_n f(t_n, x_n) = 0, \tag{3}$$

or as a one-leg (OL) method

$$\rho_n x_n - h_n f(\sigma_n t_n, \sigma_n x_n) = 0. \tag{4}$$

The familiar MS-method (3) is converted into its OL-"twin" (4) by a permutation of f and σ.

The uniform step (variable step) version of (1) is said to be stable with respect (w.r.) to $\dot{x} = \lambda x$ ($\dot{x} = \lambda(t)x$) at $q = h\lambda$ ($q_n = h_n\lambda(t_n)$) if the characteristic polynomial $\rho(\zeta) - q\sigma(\zeta)$ ($\rho_n(\zeta) - q_n\sigma_n(\zeta)$) satisfies the "root condition". For *uniform* steps, this is equivalent to boundedness of all solutions $\{x_n\}$ of the difference equation; for nonuniform grids it is a formal algebraic constraint. The set S of all q's at which the formula is stable is the stability region. The formula (1) is said to be contractive *[8] at $q(q_n)$ w.r. to a given norm $\|\bullet\|$ if $\|X_n\| \leq \|X_{n-1}\|$, $n = 1, 2, \ldots,$ where $X_n = (x_{n-k+1}, x_{n-k+2}, \ldots, x_n)$. The contractivity region $K_{\|\bullet\|}$ is the set of all q's at which the formula is contractive w.r. to $\|\bullet\|$. The formula is said to be A-stable (A-contractive) if $\overline{C}_- \subset S(\overline{C}_- \subset K_{\|\bullet\|})$; here $\overline{C}_-$ denotes the closed left half of the q-plane. The formula is said to be A_0-stable (A_0-contractive) if $\boldsymbol{R}_- \subset S(\boldsymbol{R}_- \subset K_{\|\bullet\|})$, where $\boldsymbol{R}_-$ denotes the closed negative real axis of the q-plane.

b. A_0-contractivity results.

We derived the two-parameter class of all two-step ($k = 2$) second-order ($p = 2$) formulas which are A_0-contractive w.r. to the maximum norm $\|\bullet\|_\infty$ for arbitrary step ratios $r = r_n = h_n/h_{n-1}$ [11]. This generalizes the corresponding results for uniform steps given in [8]. For the test problem $\dot{x} = \lambda(t)x$ with arbitrary $\lambda(t) \leq 0$, the OL-implementation of any variable-step A_0-contractive formula provides an A_0-stable method for arbitrary step sequences $\{h_n\}$.

c. A-contractivity results.

We also derived the one-leg parameter class of all $p = k = 2$-formulas which are A-contractive w.r. to $\|\bullet\|_\infty$ for arbitrary step ratios [11]. They are defined by

$$\begin{aligned}
\alpha_{0,n} &= -\frac{r^2}{1+r}(1-\nu), \quad \beta_{0,n} = \frac{r}{1+r} + \frac{r(r-2)}{2(1+r)}\nu - \frac{r^2}{2(1+r)}\nu^2, \\
\alpha_{1,n} &= -[1-r(1-\nu)], \quad \beta_{1,n} = \frac{1-r}{2}\nu + \frac{r}{2}\nu^2, \\
\alpha_{2,n} &= 1-\frac{r}{1+r}(1-\nu), \quad \beta_{2,n} = \frac{1}{1+r} + \frac{2r-1}{2(1+r)}\nu - \frac{r}{2(1+r)}\nu^2,
\end{aligned} \tag{5}$$

where the parameter ν satisfies

$$\frac{r-1}{r} \leq \nu \leq 1. \tag{6}$$

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*Numbers in brackets refer to the cumulative list of publications, Section 3 of this report.

For $\dot{x} = \lambda(t)x$ with arbitrary $\lambda(t)$, Re $\lambda(t) \leq 0$, the OL-implementation of any variable-step A-contractive formula provides an A-stable method for arbitrary step sequences $\{h_n\}$. We proved [11] that any formula with $p = k = 2$ is A-contractive w.r. to $\|\bullet\|_\infty$ iff it is G-stable, i.e. A-contractive w.r. to some G-norm (a norm associated with a positive definite quadratic vector form). For any given A-contractive $p = k = 2$ formula with $r = 1$ (uniform steps), specified by a parameter value $\nu = \nu_1$, $0 \leq \nu_1 \leq 1$, A-contractive extensions to non-uniform steps ($r \neq 1$) can be defined in such a way as to keep the G-norm fixed w.r. to n [11]. If this is done then, for an arbitrary step sequence $\{h_n\}$, G-stability is assured for any dissipative (monotone negative) nonlinear system.

d. Optimal A-contractive method.

An "optimally accurate" A_0-contractive p = k = 2 formula for uniform steps was obtained in [9] by minimizing a bound of the global OL-error. By minimizing the same objective function over the class of all A-contractive $p = k = 2$-formulas with $r = 1$, we found [8] the "optimal" A-contractive formula

$$-\frac{1}{6}x_{n-2}-\frac{4}{6}x_{n-1} + \frac{5}{6}x_n-h(\frac{2}{9}\dot{x}_{n-2} + \frac{2}{9}\dot{x}_{n-1} + \frac{5}{9}\dot{x}_n) = 0 \tag{7}$$

which is associated with $\nu = \nu_1 = \frac{2}{3}$. This formula was generalized [11] by extending its minimality property to the variable step case. It is defined by $\nu = 2r/(2r+1)$ and has coefficients

$$\begin{aligned} \alpha_0 &= -\frac{r^2}{(1+r)(1+2r)}, & \beta_0 &= \frac{r(1+r)}{(1+2r)^2} \\ \alpha_1 &= -\frac{1+r}{1+2r} & \beta_1 &= \beta_0 \\ \alpha_2 &= \frac{1+2r+2r^2}{(1+r)(1+2r)}, & \beta_2 &= \frac{1+2r+2r^2}{(1+2r)^2}. \end{aligned} \tag{8}$$

e. A-stability for variable steps.

Given any $p = k = 2$-formula which is A-stable for $r = 1$, we studied whether A-stability will be preserved under any perturbation of r around $r = 1$? We proved [1] that for $r = 1 + \varepsilon$ the A-stability constraints remain satisfied to first order in $|\varepsilon|$ iff the formula is A-contractive. Any other A-stable formula can be destabilized for certain $\lambda(t)$, either with increasing steps ($\varepsilon > 0$) (as is the case for the familiar two-step second-order backward diffentiation formula used in many of the popular software packages) or decreasing steps ($\varepsilon < 0$). A realization of this result using geometric step sequences $h_n = h_1 r^{n-1}$ and $\lambda(\sigma t_n) = \lambda_1 r^{1-n}$ is given in [11]. In this case, the OL-difference equation has constant coefficients and it solutions remain bounded for $n \to \infty$ iff the algebraic A-stability constraints are satisfied.

II. Josephson Inteferometer Equations

a. Background

A Josephson junction consists of two superconductors separated by an extremely thin dielectric barrier. The order parameter ϕ (phase difference between the wave functions of the tow superconductors) satisfies the well-known Josephson relations. In one space dimension, these relations, when combined with Maxwell's equations, lead to the equation

$$\wedge\phi \equiv \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial t^2} - \sigma\frac{\partial\phi}{\partial t} = k \sin \phi. \tag{9}$$

For the purpose of describing the Josephson *ac*-effect, one is seeking running solutions of (9) i.e., solutions ϕ such that $\partial\phi/\partial t$ is periodic of a certain period $(2\pi/\omega)$. In a "voltage-driven" junction, the frequency ω - which is proportional to the voltage - is prescribed and the current is given by the difference in $\frac{\partial\phi}{\partial x}$ at the two boundaries of the junction. In a "current-driven" junction, one has to find ϕ as well as its associated voltage (frequency) ω. In point junctions, the term $\frac{\partial^2\phi}{\partial x^2}$ in (9) is negligible. We have been interested in studying properties of interferometers which consist of a number of point junctions connected in various manners and loaded by currents.

b. Interferometer results

We studied the symmetric, two-junction current driven interferometer, whose equations are

$$\begin{aligned} \beta\ddot{\phi}_1 + \alpha\dot{\phi}_1 + \sin\phi_1 + \lambda^{-1}(\phi_1 - \phi_2) &= I/2 + I_c \\ \beta\ddot{\phi}_2 + \alpha\dot{\phi}_2 + \sin\phi_2 + \lambda^{-1}(\phi_2 - \phi_1) &= I/2 + I_c; \end{aligned} \tag{10}$$

here λ is proportional to the inductance, I_c is a control-current and $\phi = \omega t +$ periodic of period $2\pi/\omega$, where ω is proportional to the voltage. The following results were obtained:

i) There are subharmonic resonances in the $I-V$ curve, occurring around one-third of the resonant frequency for small λ, and around one-half of resonance for $\lambda \sim 1$. These subharmonics were previously undetected, but there is now experimental evidence of their presence.

ii) Near resonance, the periodic parts of the two phase functions are approximately sinusoidal and are of opposite signs. However, they have an almost saw-tooth shape for small voltages.

iii) The correct dependence of the resonant current on the two intrinsic parameters λ and $\Gamma \equiv \frac{1}{\sqrt{2}} \frac{\sqrt{\beta}\lambda}{\alpha}$ was found. Previously, only the approximate dependence on Γ was known.

3. CUMULATIVE LIST OF PUBLICATIONS

(under Contract F44620-75-C-0058 and its continuation, Contract F49620-77-C-0088).

1. F. Odeh and W. Liniger, "Nonlinear Fixed-*h* Stability of Linear Multistep Formulas," J. Math. Anal. Appl. **61** (1977) 691-712; and IBM Research Report RC 5717, November 11, 1975 (35 pp).

2. W. Liniger, "High-order *A*-stable Averaging Algorithms for Stiff Differential Systems," L. Lapidus and W. E. Schiesser (Eds.), *Numerical Methods for Differential Equations* Academic Press, N. Y. (1976) pp. 1-23.

3. W. Liniger and F. Odeh, "On Liapunov Stability of Nonlinear Multistep Difference Equations," IBM Research Report RC 5900, March 11, 1976 (35 pp.).

4. W. Liniger, "On Stability and Accuracy of Numerical Integration Methods for Stiff Differential Equations," IBM Research Report RC 5976, May 6, 1976 (20 pp.), and Report AFWL-TR-76-328, August 1977, pp. 36-56.

5. W. Liniger, "Stability and Error Bounds for Multistep Solutions of Nonlinear Differential Equations," Proc. ISCAS-77, Phoenix, Ariz., April 25-27, 1977, pp. 277-280, and IBM Research Report RC 6460, February 16, 1977 (6 pp.).

6. W. Liniger and F. Odeh, "A Perturbation Calculation of the *I-V* Characteristic of an Extended Josephson Junction," J. Franklin Inst. **307** (1979) 245-262 and IBM Research Report RC 7004, March 1, 1978 (20 pp.).

7. O. Nevanlinna and W. Liniger, "Contractive Methods for Stiff Differential Equations," IBM Research Report RC 7122, May 15, 1978 (79 pp.) (extended version).

8. O. Nevanlinna and W. Liniger, "Contractive Methods for Stiff Differential Equations. Part I," BIT **18** (1978) 457-474.

9. O. Nevanlinna and W. Liniger, "Contractive Methods for Stiff Differential Equations. Part II," BIT **19** (1979) 53-72.

10. W. Liniger, "Multistep and One-leg Methods for Implicit Mixed Differential Algebraic Systems," IEEE Trans. Circuits and Systems *CAS-26* (1979) 755-762, and IBM Research Report RC 7520, February 15, 1979 (22 pp.).

11. G. Dahlquist, W. Liniger and O. Nevanlinna, "Stability of Two-step Methods for Variable Integration Steps," (in preparation).

4. LIST OF PERSONNEL

Dr. Werner Liniger, Principal Investigator

Dr. Farouk Odeh, Co-principal Investigator

Dr. P. Dean Gerber, Co-principal Investigator

5. INTERACTIONS AND CONSULTATIONS

I. Lectures

1) W. Liniger, "Recent Developments in the Theory of Multistep Methods for Ordinary Differential Equations", Invited lecture, University of Toronto, Ontario, Canada, Jan. 15, 1980.

2) F. Odeh and W. Liniger, "*I-V* Characteristics and Running Solutions for 2-point-junctions Josephson Interferometers", SIAM Natl. Meeting, Alexandria, VA, June 5-7, 1980.

II. Conferences attended.

SIAM Natl. Meeting, Alexandria, VA, June 5-7, 1980 (Liniger and Odeh).

International Conference on nonlinear phenomena in Mathematical Sciences; Univ. or Texas at Arlington, June 16-20, 1980 (Odeh).

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